

# Developing Mathematical Promise **1**

*The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world.*

NCTM (1980), *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (p. 18)

**T**his need to nurture outstanding mathematical ability is even more critical today than it was over 20 years ago when the National Council of Teachers of Mathematics (NCTM) cited this necessity in their *Agenda for Action* for the 1980s. Today we realize that mathematical ability is not something that students are born with that will develop on its own. The development of mathematical potential, like any other valued ability, is something that takes dedication and hard work on the part of teachers, parents, and the students themselves. One of the goals of this book is to give teachers one more resource to help students develop their mathematical promise.

## **WHAT IS MATHEMATICAL PROMISE?**

In 1995, the NCTM appointed a Task Force on Mathematically Promising Students charged with rethinking the traditional definition of mathematically gifted students to broaden it to the more inclusive idea of mathematically promising students. In the *Report of the NCTM Task Force on Mathematically Promising Students* (Sheffield, Bennett, Beriozabal, DeArmond, & Wertheimer, 1995),

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Some of the material in this chapter is adapted from Sheffield, Linda. (February 2000). Creating and Developing Promising Young Mathematicians. *Teaching Children Mathematics*, 6(6), 416-419, 426. Copyright © 2000 by the National Council of Teachers of Mathematics. Used with permission.



mathematically promising students are defined as those who have the potential to become the leaders and problem solvers of the future. The intent was to go beyond the concept of mathematically gifted students, who traditionally had been defined as the top 3 to 5% of students based upon some standardized mathematics test. This outdated notion of mathematical giftedness frequently unnecessarily restricts access to interesting, challenging mathematics to a very small portion of the population.

The NCTM Task Force defined mathematically promising students as a function of variables that ought to be maximized. In the *Report of the NCTM Task Force*, mathematical promise is described as a function of the following:

- Ability
- Motivation
- Belief
- Experience or opportunity

The Task Force acknowledged that these variables were ones that could and should be developed in all students if we are going to maximize the numbers and levels of students with mathematical talent. This description recognizes that mathematical abilities can be enhanced and developed and are not something that some portion of the population is lacking due to some genetic deficiency. It acknowledges recent brain-functioning research that documents changes in the brain due to experiences. We know that the brain grows and develops as it responds to challenging problems, and mathematics is the perfect venue for this development

This definition also concedes that students are not always motivated to achieve at their highest possible levels, and that the popular culture in the United States may even encourage students to disguise their mathematical abilities in order to avoid negative labels such as “nerd” or “geek.” Belief in one’s ability to succeed and belief in the importance of mathematical success by the students themselves, teachers, peers, and parents are also recognized as important; lack of such beliefs, especially by females, students of color, students from lower socioeconomic groups, and students for whom English is a second language, are acknowledged as a significant barrier to learning for a large number of students.

The importance of the fourth variable, experience or opportunity to learn, is especially evident when one notes the disparity in mathematics course offerings in middle schools and high schools across the United States. In many U.S. high schools, students do not have any access to challenging mathematics courses such as Advanced Placement calculus or statistics, or if these courses are available, they are only available to a small percentage of the students in the school. This is in sharp contrast to countries such as Japan, where all students are expected to master the

basics of calculus along with other topics such as discrete mathematics, probability, statistics, geometry, and number theory in a challenging, integrated high school mathematics curriculum.

## WHO ARE MATHEMATICALLY PROMISING?

If we are to be successful in developing ever-increasing numbers of mathematically promising students, we need to be aware of some of the characteristics that these students demonstrate. Many researchers have noted that students with mathematical talent often display a mathematical frame of mind, they are able to think logically and construct generalizations, and they exhibit mathematical creativity, curiosity, and perseverance. The following list includes a few of the specific characteristics that we might look for and nurture in a mathematically promising student. Please note that not all promising students will exhibit all or even most of these characteristics. These are indicators of potential mathematically promising talent that should be developed as much as possible in all students. Teachers and parents should strive to find interesting mathematical challenges at all levels that will engage students in this development of mathematical power.

### ***CHARACTERISTICS OF A MATHEMATICALLY PROMISING STUDENT***

#### **MATHEMATICAL FRAME OF MIND**

1. Loves exploring patterns and puzzles
2. Sees mathematics and structure in a variety of situations
3. Recognizes, creates, and extends patterns
4. Organizes and categorizes information
5. Has a deep understanding of simple mathematical concepts, including a strong number sense

#### **MATHEMATICAL FORMALIZATION AND GENERALIZATION**

1. Generalizes the structure of a problem, often from only a few examples
2. Uses proportional reasoning
3. Thinks logically and symbolically with quantitative and spatial relations
4. Develops proofs and other convincing arguments



### **MATHEMATICAL CREATIVITY**

1. Processes information flexibly—switches from computation to visual to symbolic to graphic representations as appropriate in solving problems
2. Reverses processes—can switch from a direct to a reverse train of thought
3. Has original approaches to problem solving—solves problems in unique ways, tries unusual methods
4. Strives for mathematical elegance and clarity in explaining reasoning

### **MATHEMATICAL CURIOSITY AND PERSEVERANCE**

1. Is curious about mathematical connections and relationships—asks “why” and “what if”
2. Has energy and persistence in solving difficult problems
3. Digs beyond the surface of a problem, continues to explore after the initial problem has been solved

The following characteristics may be useful in a mathematics class but are not necessary for a student to be mathematically promising:

1. Speed and accuracy with computation
2. Memory for formulas and facts
3. Spatial ability

Sheffield, Fall 2000, pp. 7–8. Used with permission of the Kentucky Association for Gifted Education.

In this book, you will find numerous examples of problems and questions designed to encourage and enhance students’ mathematical promise. It is hoped that together with the students, you will create many other intriguing problems and puzzles for yourselves.

## WHAT ARE THE GOALS OF MATHEMATICS INSTRUCTION?

Suggestions for ways in which teachers, parents, and students can work together to increase levels of mathematical power and proficiency are given throughout this book. The intent is to move students along a continuum such as the following:



innumerates    doers    computers    consumers    problem solvers    problem posers    creators

This model notes that some students are virtually innumerate, lacking even basic concepts of number and computation. Slightly above these are students who can do some mathematics or who are good at basic computation, but who do not apply these concepts to everyday problems. Students who are wise consumers are better able to function in our increasingly technological world, but this is not sufficient. In the 1980 NCTM *Agenda for Action*, problem solving was noted as a major goal for the end of the 20th century. However, even this is not enough for the problems that we face in the 21st century. Students today must be able to not only solve the problems that others have outlined but must also be able to recognize and pose the critical problems to solve tomorrow. We cannot tell students just what problems they will face in the future. They must learn to define the problems and create new mathematics with which to tackle them. These skills can begin to be developed as early as preschool, and this development should continue throughout one's life.

## HOW MIGHT WE FIND AND/OR CREATE GOOD PROBLEMS TO EXPLORE?

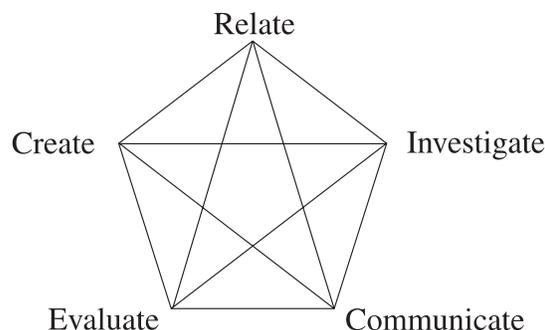
If we wish students to become proficient problem solvers and to move from there to becoming the posers of interesting mathematical problems and the creators of new mathematics, we must provide them with multiple opportunities to work on interesting mathematical investigations. The following is a list of criteria for recognizing and developing this type of investigation or problem:

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1. Tasks should ask questions that make students think, not questions that make them guess what the teacher is thinking.
  2. Tasks should enable children to build on previous knowledge and to discover previously unknown mathematical principles and concepts.
  3. Tasks should be rich, with a wide range of opportunities for children to explore, reflect, extend, and branch out into new related areas.
  4. Tasks should give children the opportunity to demonstrate abilities on a variety of levels and in a variety of ways, verbally, geometrically, graphically, algebraically, numerically, and so on. Try scaffolding the questions, so every child can be successful on some level while maintaining a high level of challenge for children who are ready and eager to progress. Scaffolding means that a problem might have several parts, beginning with a relatively simple question that all children should have success answering and then building to more complex questions that challenge even the most skillful students. In this way, all students can be working on the same basic problem at an appropriate level, much in the same way that students might all be writing a story on the same theme but writing it on widely different levels.
  5. Tasks should allow children to use their abilities to question, reason, communicate, solve problems, and make connections to other areas of mathematics as well as to other subject areas and real world problems.
  6. Tasks should make full, APPROPRIATE use of technology such as calculators and computers as well as mathematical manipulatives and models.
  7. Tasks should give time for individual reflection and problem solving as well as time for group exploration and discovery.
  8. Tasks should be interesting and should actively involve the child.
  9. Tasks should be open, with more than one right answer and/or more than one path to solution.
  10. Tasks should encourage continued exploration once the initial question has been answered. One problem should be a springboard for several others. Teachers should work on these problems with colleagues before trying them with students to see how many solutions, patterns, generalizations, and related problems they can find themselves.

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**Figure 1.1.** Open-Approach Heuristic

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## WHAT MODELS MIGHT WE USE TO INCREASE THE NUMBERS AND LEVELS OF MATHEMATICALLY PROMISING STUDENTS?

One way to begin to develop this mathematical potential is to teach students to use an open-approach heuristic such as the one shown in Figure 1.1.

In this model, students might start anywhere and proceed in a non-linear fashion to creatively investigate a problem. For example, a student might relate ideas about solving this problem to previous problems that they have solved, investigate those ideas, create new problems to work on, evaluate solutions, communicate the results, and think of other related problems to work on. Problems suggested in this book will show students ways to delve deeply into interesting problems in this manner.

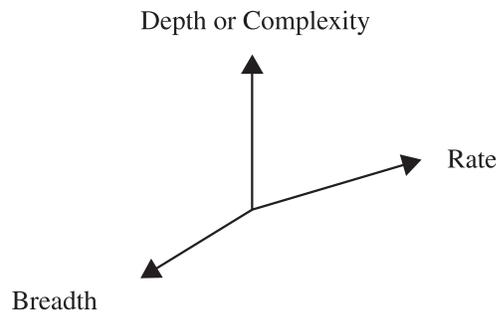
Note that in this model, students do not stop when they have found a solution. Too often, students are satisfied with getting an answer to a problem and not looking at it any further. In this way, they miss the excitement of thinking deeply about mathematical ideas and discovering new concepts. Students need to learn to explore problems to find that the fun has only just begun when the original problem has been solved. Mathematicians will tell you that the real mathematics begins after a solution has been found.

This model gets away from the traditional question of mathematics for gifted and talented students that asked whether students' mathematical experiences should be enriched (often meaning adding additional topics to the curriculum) or accelerated (often meaning going through the traditional curriculum more quickly).

Following the Third International Mathematics and Science study, one frequently heard comments about the mathematics curriculum in the United States as being "an inch deep and a mile wide" (Schmidt, McKnight, & Raizen, 1996). Mathematics textbooks in the United States



**Figure 1.2.** Model for Adding Depth and Complexity to the Mathematics Curriculum



tend to cover large numbers of topics at a relatively shallow level and repeat the same topics for years. For example, it is not uncommon to see children studying whole number addition with regrouping each year from first grade through sixth grade (and sometimes even later). Whole number addition is often one of a hundred or more topics that students encounter in a mathematics class each year. Given the already large number of topics and the shallow level of coverage, it makes more sense to think of a mathematical model of instruction that is at least three dimensional, such as that shown in Figure 1.2.

In this model, an optimal level should be found for all three dimensions, but it is more critical to add depth and complexity to the study of mathematics rather than to focus on the addition of “enrichment” topics to broaden the curriculum or to increase the rate of instruction at the expense of depth. Many students seem to think of mathematics as a topic to finish as quickly as possible rather than one to enjoy and savor. As I hope you will experience as you work through the problems in this book, mathematics is much more enjoyable when you learn to follow the motto of Professor Arnold Ross of Ohio State: “Think deeply about simple things.” Professor Ross has worked with many of the greatest mathematical thinkers in the United States at the high school, college, and graduate level for over 60 years. His greatest legacy may be that he has taught students that the most complex and enjoyable mathematical explorations begin with simple concepts that they mine for their richness and elegance.

Problems presented in this book will often use some of the following questions to help students learn to explore problems in depth.

### Organization and Representation

1. How might I represent, simulate, model, or visualize these ideas in various ways?

2. How might I sort, organize, and present this information?
3. What patterns do I see in this data?

### **Rules and Procedures**

1. What steps might I follow to solve that? Are they reversible?  
Is there an easier or better way?
2. Do I have enough information? Too much information?  
Conflicting information?

### **Optimization and Measurement**

1. How big is it?
2. What is the largest possible answer? The smallest?
3. How many solutions are possible? Which is the best?
4. What are the chances? What is the best chance?

### **Reasoning and Verification**

1. Why does that work? If it does not work, why not?
2. Will that always work? Will that ever work?
3. Is that reasonable? Can you prove that? Are you sure?

As students begin to explore problems in depth, they should also realize that answering the original question is just the beginning. The real mathematics often starts after the original question has been answered. Creative mathematicians learn to question the answers; they don't just answer the questions. In this book, to encourage students to become problem posers and creators of new mathematics, they will be encouraged to ask several of the following questions:

### **Generalizations**

1. What other patterns do I notice?
2. Can I generalize these patterns?
3. Are there exceptions to my rules? Under what conditions does this work or not work?

### **Comparisons and Relationships**

1. How is this like other mathematical problems or patterns that I have seen?
2. How does it differ? What other questions does this raise?
3. How does this relate to real-life situations or models?



4. How are two factors or variables related? What new relationships can I find?
5. What if I change one or more parts of the problem? How does that affect the outcomes? (Adapted from Sheffield, February 2000, p. 419)

With questions such as these, the solution to the original problem is used as a springboard to deeper, more original mathematical thinking.

## HOW SHOULD WE ASSESS SUCCESS?

If we wish students to become problem posers and creators of mathematics, we must change our traditional methods of assessment. This type of mathematical exploration cannot be evaluated using a multiple choice or fill-in-the-blank exam. Portfolios of student work that show progress over the course of a semester or an academic year are often a much better indicator of a student's development of mathematical power. We must also let students know that we are looking for depth of reasoning and mathematical creativity. Students themselves can help in the development of specific scoring rubrics for their problems and in analyzing and evaluating each other's work. The following are examples of criteria that might be used for this assessment:

1. Depth of understanding: the extent to which core concepts are explored and developed (this should be related to national, state, and/or local curriculum goals and objectives)
2. Fluency: the number of different correct answers, methods of solution, or new questions formulated
3. Flexibility: the number of different categories of answers, methods, or questions, such as numeric, algebraic, geometric, or graphical
4. Originality: solutions, methods, or questions that are unique and show insight
5. Elaboration or elegance: quality of expression of thinking, including charts, graphs, drawings, models, and words
6. Generalizations: patterns that are noted, hypothesized, and verified for larger categories
7. Extensions: related questions that are asked and explored, especially those involving why and what if

You may wish to use a rubric such as the one shown in Table 1.1 to evaluate some of the students' major mathematics projects.

**TABLE 1.1** Problem Posing and Creativity

Assessment Criteria	Scores			
	1	2	3	4
Depth of Understanding	Little or no understanding	Partial understanding; minor mathematical errors	Good understanding; mathematically correct	In-depth understanding; well-developed ideas
Fluency	One incomplete or unworkable approach	At least one appropriate approach or related question	At least two appropriate approaches or good related questions	Several appropriate approaches or new related questions
Flexibility		All approaches use the same method (e.g., all graphs, all algebraic equations, etc.)	At least two methods of solution (e.g., geometric, graphical, algebraic, physical modeling)	Several methods of solution (e.g., geometric, graphical, algebraic, physical modeling)
Originality	Method may be different but does not lead to a solution	Method will lead to a solution but is fairly common	Unusual, workable method used by only a few students	Unique, insightful method used only by one or two students
Elaboration or Elegance	Little or no appropriate explanation given	Explanation is understandable but may be unclear in some places	Clear explanation using correct mathematical terms	Clear, concise, precise explanations making good use of graphs, charts, models, or equations
Generalizations and Reasoning	No generalizations made, or they are incorrect and reasoning is unclear	At least one correct generalization made; may not be well supported with clear reasoning	At least one well-made, supported generalization, or more than one correct but unsupported generalization	Several well-supported generalizations; clear reasoning
Extensions	None included, or extensions are not mathematical	At least one related mathematical question appropriately explored	One related question explored in depth, or more than one appropriately explored	More than one related question explored in depth

SOURCE: Sheffield (February 2000, p. 419). Used with permission of the National Council of Teachers of Mathematics. © 2000.



## STRUCTURE OF THE ACTIVITIES IN THIS BOOK

In Chapters 2 through 5, you will find a variety of activities designed to encourage students to explore each of the content strands from the NCTM *Principles and Standards for School Mathematics* (2000):

- Chapter 2: Number and Operations
- Chapter 3: Algebra
- Chapter 4: Geometry and Measurement
- Chapter 5: Data Analysis and Probability

In each of these chapters, you will find activities suggested for Level A (generally novices or students from prekindergarten through Grade 2), for Level B (generally intermediate students who have some experience with the Level A investigation or students in Grades 3 through 5), and for Level C (generally students who have mastered the Level A and Level B investigations and are ready to move on to more sophisticated generalizations, perhaps using algebra, or students in Grades 6 through 8). These grade levels are simply a suggestion for a starting point. As you work through these activities with the students, you may find that students at one of the upper-grade levels do not have the prerequisite understanding to begin at the suggested level for that grade, and you may wish to begin with activities that have been suggested for an earlier level. You probably also will find that some of your students are capable of going far beyond the suggested activities for a given grade level. Feel free to adapt the activities to meet the needs of the students in your own classroom. You should develop probing questions to follow up on new student discoveries and encourage the students to do the same as they learn to investigate problems on a deeper level.

Activities have been designed to encourage the students to become problem posers and creators of mathematics. You should introduce the open problem-solving heuristic (see Figure 1.1) to students. You will find a reproducible form of this (Form 1.1) at the end of the chapter that you may wish to make into a transparency for an overhead projector or enlarge for a poster to post where students can be reminded of this process. Please remember that even though the activities are introduced in a fixed order, actual student explorations may proceed in very different orders as students investigate questions of interest to themselves.

Start at any point on the diagram and proceed in any order.

- Relate the problem to other problems that you have solved. How is this similar to other mathematical ideas that you have seen? How is it different?
- Investigate the problem. Think deeply and ask questions.

- Evaluate your findings. Did you answer the question? Does the answer make sense?
- Communicate your results. How can you best let others know what you have discovered?
- Create new questions to explore. What else would you like to find out about this topic? Start a new investigation.

You may need to simplify some of the language for very young children when you first begin this process, but even very young children have a strong natural curiosity that will aid them in the development of this mathematically powerful technique.

You should also copy Form 1.2, Questions! Questions! Questions! at the end of this chapter for the students and post it in the classroom. This will remind them of questions to ask as they investigate the problems. Once students are proficient at this, they should begin to investigate many ideas that you as a teacher may not have explored before. Don't be alarmed if they ask questions that you cannot answer. This is a great sign that students are indeed learning to "think deeply." If you and the students get stuck in an investigation, don't forget that there are often great resources available as nearby as your computer. At the end of the book, you will find a list of resources for additional problems and for "expert" mathematicians who are willing to help you in your investigations.

Each of the investigations in Chapters 2 through 5 will be presented in the same format with the following sections:

*Relate.* Here the stage is set for the investigation by connecting it to the NCTM *Principles and Standards* and prior mathematical learning. Mention is also made here of routine exercises that may be encountered in a traditional mathematics program that address these same topics on a less challenging level. These investigations may often be substituted for those parts of the textbook, thus freeing more time for complex, thought-provoking explorations.

*Investigate.* This is the initial problem to start the students thinking about the investigation. There will be a reproducible form to accompany this problem that can be copied and given to the students. This section also will include suggestions for the teacher that will be needed to get started.

*Evaluate and Communicate.* This section includes ideas for the students to evaluate their own thinking and suggestions for probing assessment questions and responses for the teacher. It also includes ideas for the students for sharing their solutions. Correct answers with possible student discussion are included in this section.



*Create.* Ideas for extending and deepening the investigation are given here. These ideas should help even the most accomplished students to add depth and complexity to their reasoning.

*Discussion.* This section gives teachers hints on what to look for in students' solutions and ideas for encouraging them to dig more deeply into the mathematical concepts presented.

In the Resources, you will find a list of resources for other ideas for extending and challenging the thinking of mathematically promising students. These are designed to help you and the students continue your investigations and strengthen your mathematical power and enjoyment. Many of these make use of resources on the ever-changing World Wide Web. If students do not have access to these resources at home, you may need to help them find a way to use other public access to computers and the Internet either at school or in a public library. You also may wish to have a parents' mathematics night to explain your mathematical goals and processes and to let parents share in exploring the rich resources available.

## REPRODUCIBLE FORMS

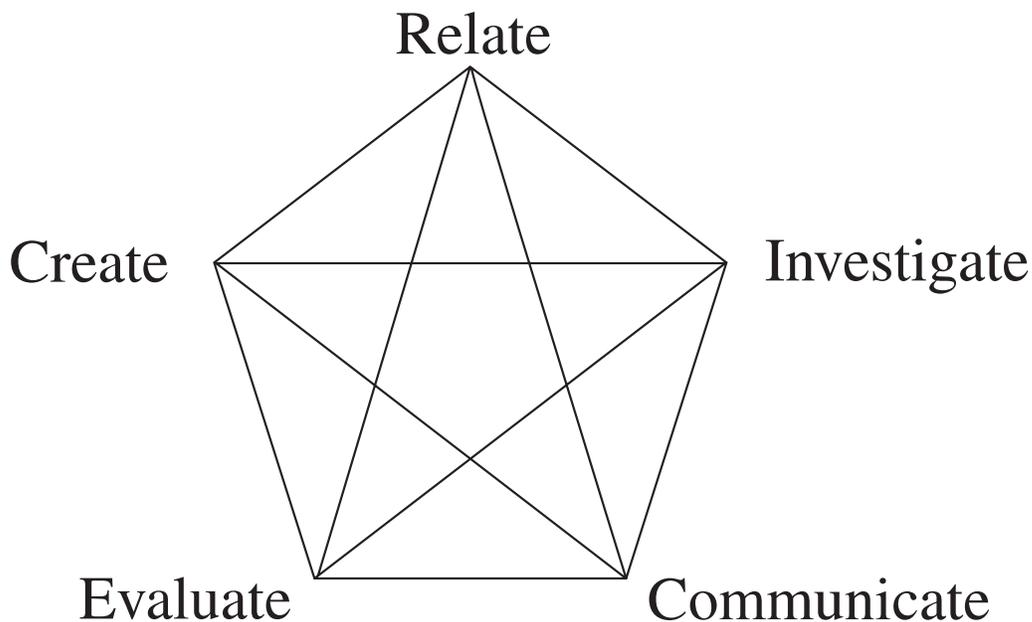
At the end of each chapter, you will find forms that you are permitted to reproduce for the students in your classroom. These forms should make it easy for you as a teacher to present the problems to the students.

As you begin your own explorations through this book, be sure to take time yourself to enjoy the thought processes involved in investigating new ideas. Let the students see that none of us have all the answers, but we can all take pleasure in the search for them.

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**Form 1.1. Open Approach to Problem Solving**

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Start at any point on the diagram and proceed in any order.

- Relate the problem to other problems that you have solved. How is this similar to other mathematical ideas that you have seen? How is it different?
- Investigate the problem. Think deeply and ask questions.
- Evaluate your findings. Did you answer the question? Does the answer make sense?
- Communicate your results. How can you best let others know what you have discovered?
- Create new questions to explore. What else would you like to find out about this topic? Start a new investigation.



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**Form 1.2. Questions! Questions! Questions!**

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**Think Deeply About Simple Things**

- How might I model or organize my thoughts?
- Why did that work?
- Why did that not work?
- How is this like any other problem I have solved?
- How is this different from other problems?
- Is that always true?
- Will that ever work?
- What patterns do I notice?
- What is the largest possible answer? The smallest?
- How many solutions are possible?
- How might I best convince others of my results?

**Question the Answers: Don't Just Answer the Questions**

- What other questions came up as I solved the original problem?
- What if I changed part of the problem?
- What if part of the problem were not there or a new part were added?
- Can I do that another way? How many ways might I . . . ?
- What other patterns do I notice?
- What generalizations might I make? Are they always true?
- What other problems might I solve in a similar way?