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Mathematics Assessment Probes

To differentiate instruction effectively, teachers need diagnostic assessment strategies to gauge their students' prior knowledge and uncover their understandings and misunderstandings. By accurately identifying and addressing misunderstandings, teachers prevent their students from becoming frustrated and disenchanted with mathematics, which can reinforce the student preconception that "some people don't have the ability to do math." Diagnostic strategies also allow for instruction that builds on individual students' existing understandings while addressing their identified difficulties. The Mathematics Assessment Probes in this book allow teachers to target specific areas of difficulty as identified in research on student learning. Targeting specific areas of difficulty—for example, the transition from reasoning about whole numbers to understanding numbers that are expressed in relationship to other numbers (decimals and fractions)—focuses diagnostic assessment effectively (National Research Council, 2005, p. 310).

Mathematics Assessment Probes represent one approach to diagnostic assessment. The probes specifically elicit prior understandings and commonly held misconceptions that may or may not have been uncovered during an instructional unit. This elicitation allows teachers to make instructional choices based on the specific needs of students. Examples of commonly held misconceptions elicited by a Mathematics Assessment Probe include ideas such as "an equals sign means *the answer follows*" and "to add fractions, add the numerators and then add the denominators." It is important to make the distinction between what we might call a silly mistake and a more fundamental one, which may be the product of a deep-rooted misunderstanding. It is not uncommon for different students to display the same misunderstanding every year. Being aware of and eliciting common misunderstandings and drawing students' attention to them can be a valuable teaching technique (Griffin & Madgwick, 2005) that should be used no matter what particular curriculum program a teacher uses, be it commercial, district developed, or teacher developed.

The process of diagnosing student understandings and misunderstandings and making instructional decisions based on that information is the key to increasing students' mathematical knowledge.

To use the Mathematics Assessment Probes for this purpose, teachers need to

- determine a question;
- use a probe to examine student understandings and misunderstandings;
- use links to cognitive research, standards, and math education resources to drive next steps in instruction;
- implement the instructional unit or activity; and
- determine the impact on learning by asking an additional questions.

The probes and the above process are described in detail in this chapter. The Teacher Notes that accompany each of the Mathematics Assessment Probes in Chapters 3 through 6 include information on research findings and instructional implications relevant to the instructional cycle described above. Detailed information about the information provided in the accompanying Teacher Notes is also described in detail within this chapter.

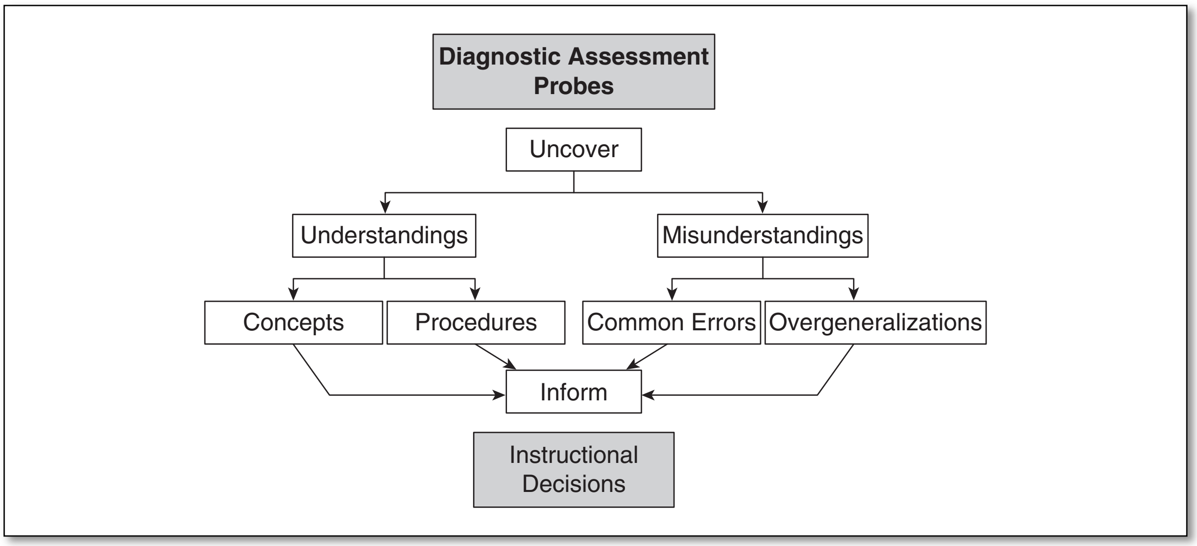
WHAT TYPES OF UNDERSTANDINGS AND MISUNDERSTANDINGS DOES A MATHEMATICS ASSESSMENT PROBE UNCOVER?

Developing understanding in mathematics is an important but difficult goal. Being aware of student difficulties and the sources of those difficulties, and designing instruction to diminish them, are important steps in achieving this goal (Yetkin, 2003). The Mathematics Assessment Probes are designed to uncover student understandings and misunderstandings; the results are used to inform instruction rather than make evaluative decisions. As shown in Figure 1.1, the understandings include both conceptual and procedural knowledge, and misunderstandings can be classified as common errors or overgeneralizations. Each of these is described in the following in more detail.

Understandings: Conceptual and Procedural Knowledge

> Research has solidly established the importance of conceptual understanding in becoming proficient in a subject. When students understand mathematics, they are able to use their knowledge flexibly. They combine factual knowledge, procedural facility, and conceptual understanding in powerful ways. (National Council of Teachers of Mathematics [NCTM], 2000)

Figure 1.1 Diagnostic Assessment Probes



Source: Rose, C., Minton, L. & Arline, C. (2007).

Conceptual Understanding

Students demonstrate conceptual understanding in mathematics when they

- recognize, label, and generate examples and nonexamples of concepts;
- use and interrelate models, diagrams, manipulatives, and so on;
- know and apply facts and definitions;
- compare, contrast, and integrate concepts and principles;
- recognize, interpret, and apply signs, symbols, and terms; and
- interpret assumptions and relationships in mathematical settings.

Procedural Knowledge

Students demonstrate procedural knowledge in mathematics when they

- select and apply appropriate procedures;
- verify or justify a procedure using concrete models or symbolic methods;
- extend or modify procedures to deal with factors in problem settings;
- use numerical algorithms;
- read and produce graphs and tables;
- execute geometric constructions; and
- perform noncomputational skills, such as rounding and ordering.

Source: From U.S. Department of Education, 2003, Chapter 4.

The relationship between understanding concepts and being proficient with procedures is complex. The following description gives an example of how the Mathematics Assessment Probes elicit conceptual or procedural understanding. The What Is the Value of the Digit? probe (see Figure 1.2) is designed to elicit whether students understand place value beyond being able to procedurally connect numbers to their appropriate places.

Students who choose B, There is a 2 in the ones place, and E, There is a 1 in the tenths place, understand the value of the place of digits within a number. By also choosing C, There are 21.3 tenths, and H, There are 213 hundredths, these students also demonstrate a conceptual understanding of the relationship between the value of the places and the number represented by the specific combination of digits.

Figure 1.2 What Is the Value of the Digit? (see page 71, Probe 6, for more information on this probe)

Teacher prompt: "Circle all of the statements that are true for the number 2.13."

Statement	Explanation (why circled or not circled)
A) There is a 3 in the ones place.	
B) There is a 2 in the ones place.	
C) There are 21.3 tenths.	
D) There are 13 tenths.	
E) There is a 1 in the tenths place.	
F) There is a 3 in the tenths place.	
G) There are 21 hundredths.	
H) There are 213 hundredths.	

Misunderstandings: Common Errors and Overgeneralizations

In *Hispanic and Anglo Students' Misconceptions in Mathematics,* Jose Mestre (1989) describes misconceptions as follows:

> Students do not come to the classroom as "blank slates." Instead, they come with theories constructed from their everyday experiences. They have actively constructed these theories, an activity crucial to all successful learning. Some of the theories that students use to make sense of the world are, however, incomplete half-truths. They are misconceptions.
>
> Misconceptions are a problem for two reasons. First, they interfere with learning when students use them to interpret new experiences. Second, students are emotionally and intellectually attached to their misconceptions because they have actively constructed them. Hence, students give up their misconceptions, which can have such a harmful effect on learning, only with great reluctance. (para. 2–3)

For the purposes of this book, these misunderstandings or misconceptions will be categorized into common errors and overgeneralizations, which are described in more detail below.

Common Error Patterns

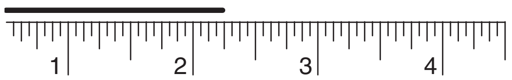
Common error patterns refer to systematic uses of inaccurate and inefficient procedures or strategies. Typically, this type of error pattern indicates nonunderstanding of an important math concept (University of Kansas, 2005). Examples of common error patterns include consistent misuse of a tool or steps of an algorithm, such as an inaccurate procedure for computing or the misreading of a measurement device. The following description gives an example of how the Mathematics Assessment Probes elicit common error patterns.

One of the ideas the What's the Measure? probe (see Figure 1.3) is designed to elicit is the understanding of zero point. "A significant minority of older children (e.g., fifth grade) respond to nonzero origins by simply reading off

Figure 1.3 What's the Measure? Probe (See page 180, Probe 23, for more information on this probe.)

Use the measuring tool provided to measure the length of the line.

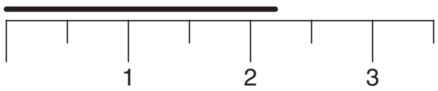
A)



Is this line approximately $2\frac{1}{4}$ units long?____

Explain why or why not.

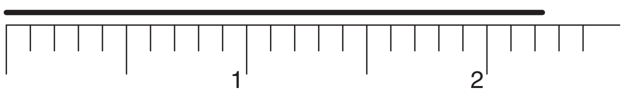
B)



Is this line approximately $2\frac{1}{4}$ units long?____

Explain why or why not.

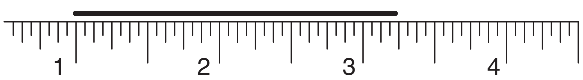
C)



Is this line approximately $2\frac{1}{4}$ units long?____

Explain why or why not.

D)



Is this line approximately $2\frac{1}{4}$ units long?____

Explain why or why not.

whatever number on a ruler aligns with the end of the object (Lehrer et al., 1998a)" (NCTM, 2003, p. 183). We have found many middle school students also make this same mistake.

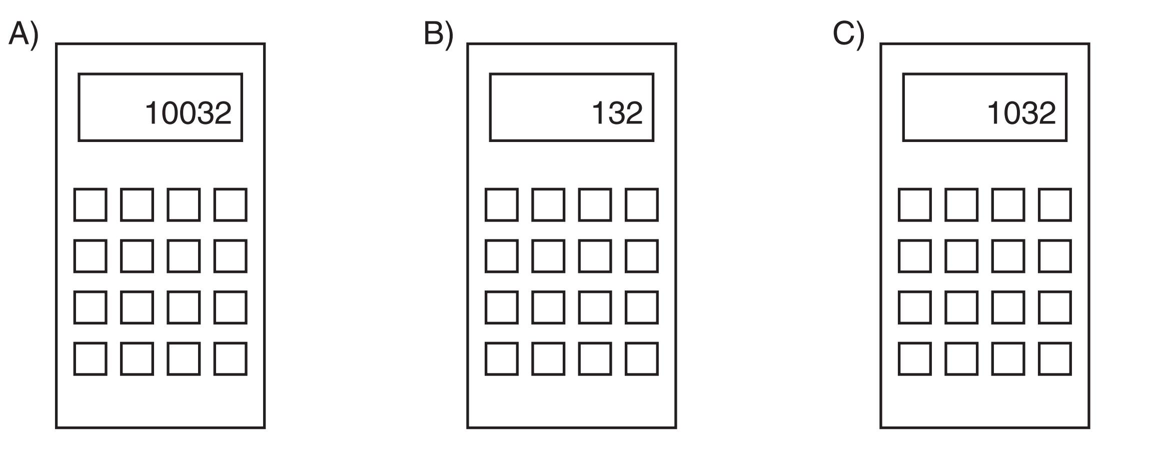
The correct answers are A, no; B, yes; C, yes; and D, yes. Students who *include* A typically do not consider the nonzero starting point and give the length as the number on the ruler aligned to the endpoint of the segment. Students who exclude D are not considering the nonzero starting point.

Overgeneralizations

Often, students learn an algorithm, rule, or shortcut and then extend this information to another context in an inappropriate way. These misunderstandings are often overgeneralizations from cases that students have seen in prior instruction (Griffin & Madgwick, 2005). To teach in a way that avoids creating any misconceptions is not possible, and we have to accept that students will make some incorrect generalizations that will remain hidden unless the teacher makes specific efforts to uncover them (Askew & Wiliam, 1995).The following example illustrates how the Mathematics Assessment Probes can elicit overgeneralizations. The What is the Number? probe (see Figure 1.4) is designed to elicit the overgeneralizations students make from the way a number

Figure 1.4 What's the Number? (See page 49, Probe 2, for more information on this probe.)

Teacher prompt: "Circle the calculator that shows the number *one hundred thirty-two*."



Directions:

Show the calculators above to the student, and ask her to circle the calculator that shows the number you say, *"One hundred thirty-two."*

"How did you decide on calculator _____?"

Record reasoning:

is read orally to the numeric form. Students who incorrectly choose A often disregard place value concepts because of this overgeneralization.

A single probe can elicit both common errors and overgeneralizations depending on how individual students respond to the particular question. In addition to uncovering common misunderstandings, the Mathematics Assessment Probes also elicit *uncommon* misconceptions that may not be uncovered and could continue to cause difficulty in understanding a targeted concept.

HOW WERE THE MATHEMATICS ASSESSMENT PROBES DEVELOPED?

Developing an assessment probe is different from creating appropriate questions for summative quizzes, tests, or state and national exams. The probes in this book were developed using the process described in *Mathematics Curriculum Topic Study: Bridging the Gap Between Standards and Practice* (Keeley & Rose, 2006).

The process is summarized as follows:

- **Identify the topic you plan to teach, and use national standards to examine concepts and specific ideas related to the topic.** The national standards used to develop the probes for this book were NCTM's (2000) *Principles and Standards for School Mathematics* and the American Association for the Advancement of Science's (AAAS, 1993) *Benchmarks for Science Literacy*.

- **Select the specific concepts or ideas you plan to address, and identify the relevant research findings.** The source for research findings include NCTM's (2003) *Research Companion to Principles and Standards for School Mathematics*, Chapter 15 of AAAS's (1993) *Benchmarks for Science Literacy,* and additional supplemental articles related to the topics of the probes.

- **Focus on a concept or a specific idea you plan to address with the probe, and identify the related research findings.** Choose the type of probe structure that lends itself to the situation (see more information on probe structure following the Gumballs in a Jar example on page 9). Develop the stem (the prompt), key (correct response), and distracters (incorrect responses derived from research findings) that match the developmental level of your students.

- **Share your assessment probes with colleagues for constructive feedback, pilot with students, and modify as needed.**

Figure 1.5, which is taken from Keeley and Rose (2006), provides the list of concepts and specific ideas related to the probability of simple events. The shaded information was used as the focus in developing the probe Gumballs in a Jar (see Figure 1.6).

Figure 1.5 Probability Example

Topic: Probability (Simple Events)

Concepts and Ideas	*Research Findings*
• Events can be described in terms of Being more or less likely, impossible or certain (Grades 3–5, AAAS, 1993, p. 228). • Probability is the measure of the likelihood of an event and can be represented by a number from 0 to 1 (Grades 3–5, NCTM, 2000, p. 176). • Understand that 0 represents the probability of an impossible event and 1 represents the probability of a certain event (Grades 3–5, NCTM, 2000, p. 181). • Probabilities are ratios and can be expressed as fractions, percentages, or odds (Grades 6–8, AAAS, 1993, p. 229). • Methods such as organized lists, tree diagrams, and area models are helpful in finding the number of possible outcomes (Grades 6–8, NCTM, 2000, pp. 254–255). • The theoretical probability of a simple event can be found using the ratio of a favorable outcome to total possible outcomes (Grades 6–8, AAAS, 1993, p. 229). • The probability of an outcome can be tested with simple experiments and simulation (NCTM, 2000, pp. 254–255). • The relative frequency (experimental probability) can be computed using data generated from an experiment or simulation (Grades 6–8, NCTM, 2000, pp. 254–255). • The experimental and theoretical probability of an event should be compared with discrepancies between predictions and outcomes from a large and representative sample taken seriously (Grades 6–8, NCTM, 2000, pp. 254–255).	**Understandings of Probability (NCTM, 2003, pp. 216–223)** • Lack of understanding of ratio leads to difficulties in understanding of chance. • Students tend to focus on absolute rather than relative size. • Although young children do not have a complete understanding of ratio, they have some intuitions of chance and randomness. • A continuum of probabilistic thinking includes subjective, transitional, informal, quantitative, and numerical levels. • Third grade (approx) is an appropriate place to begin systematic instruction. • "Equiprobability" is the notion that all outcomes are equally likely, disregarding relative and absolute size. • The outcome approach is defined as the misconception of predicting the outcome of an experiment rather than what is likely to occur. A typical response to questions is, "Anything can happen." • Intuitive reasoning may lead to incorrect responses. Categories include representativeness and availability. • Wording to task may influence reasoning. NAEP results show fourth and eighth graders have difficulty with tasks involving probability as a ratio of "m chances out of n" but not with "1 chance out of n" (NCTM, 2003, p. 222). • Increased understanding of sample space stems from multiple opportunities to determine and discuss possible outcomes and predict and test using simple experiments. **Uncertainty (AAAS, 1993, p. 353)** • Upper-elementary students can give correct examples for certain, possible and impossible events, but they have difficulties calculating the probability of independent and dependent events. • Upper-elementary students create "part to part" rather than "part to whole" relationships.

From Keeley, P., & Rose, C. M. (2006).

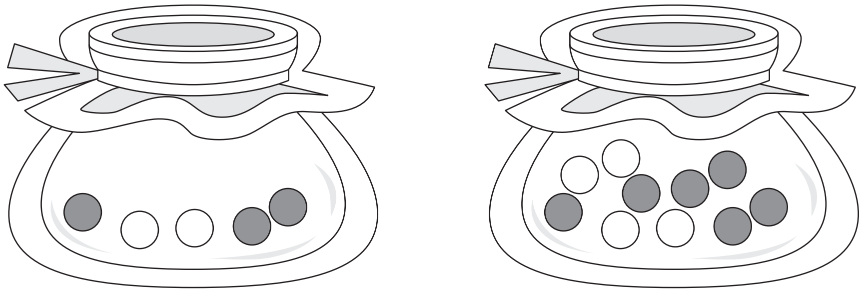
Figure 1.6 Gumballs in a Jar

Two jars both contain black and white gumballs.

Jar A: 3 black and 2 white

Jar B: 6 black and 4 white

Which response best describes the chance of getting a black gumball?



A. There is a better chance of getting a black gumball from Jar A.

B. There is a better chance of getting a black gumball from Jar B.

C. The chance of getting a black gumball is the same for both Jars A and B. Explain your reasons for the answer you selected.

From Keeley, P., & Rose, C. M. (2006).

The probe is used to reveal common errors regarding probability, such as focusing on absolute size, or a lack of conceptual understanding of probability as a prediction of what is likely to happen. There is the same chance you will pick a black gumball out of each jar. Jar A has a probability of $\frac{3}{5}$, and Jar B has a probability of $\frac{6}{10} = \frac{3}{5}$. There are a variety of trends in correct thinking related to this probe, some of which are doubling, ratios, and percents. Some students might correctly choose answer C but use incorrect reasoning, such as "You can't know for sure since anything can happen," an explanation that indicates a lack of conceptual understanding of probability. Other students may demonstrate partial understanding with responses such as "each jar has more black than white." Some students reason that there are fewer white gumballs in Jar A compared to Jar B and therefore there is a better chance of picking a black gumball from Jar A. Others observe that Jar B has more black gumballs compared to Jar A and therefore reason that there is a better chance of picking a black gumball. In both cases, students are focusing on absolute size instead of relative size in comparing the likelihood of events. Students sometimes choose Distracter A due to an error in counting or calculation.

Additional probes can be written using the same list of concepts and specific ideas related to the probability of simple events. For example, by focusing on the statement from the research, "NAEP results show fourth and eighth graders have difficulty with tasks involving probability as a ratio of '*m* chances out of *n*' but not with '1 chance out of *n*'" (NCTM, 2003, p. 222), a probe using an example of each can diagnose if students are demonstrating this difficulty.

WHAT IS THE STRUCTURE OF A MATHEMATICS ASSESSMENT PROBE?

A probe is a cognitively diagnostic paper-and-pencil assessment developed to elicit research-based misunderstandings related to a specific mathematics topic. The individual probes are designed to be (1) *easy to use* and copy ready for use with students; (2) *targeted* to one mathematics topic for short-cycle intervention purposes; and (3) *practical,* with administration time targeted to approximately 5 to 15 minutes.

Each one-page probe consists of selected response items (called Tier 1) and explanation prompts (called Tier 2), which together elicit common understandings and misunderstandings. Each of the tiers is described in more detail below.

Tier 1: Elicitation

As the elicitation tier is designed to uncover common understandings and misunderstandings, a structured format using a question or series of questions followed by correct answers and incorrect answers (often called distracters) is used to narrow ideas to those found in the related cognitive research. The formats typically fall into one of seven categories.

1. Selected response. One question, one correct answer, and several distracters (see Figures 1.7 and 1.8).

Figure 1.7 Variation: Crayon Count (See page 90, Probe 8A, for more information about this probe.)

There is a box of orange, blue, and purple crayons. Half ($\frac{1}{2}$) of the crayons are purple. Some crayons are orange, and a quarter ($\frac{1}{4}$) of the crayons are blue.

How many of the crayons are orange?

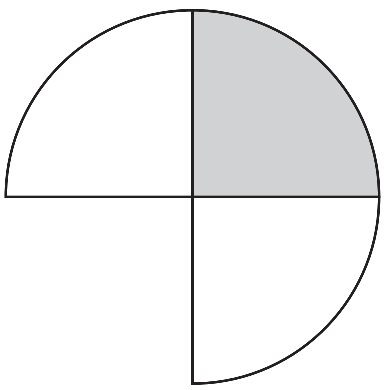
A) $\frac{3}{4}$

B) $\frac{2}{4}$

C) $\frac{1}{4}$

Figure 1.8 Variation: How Much Is Shaded? (See page 98, Probe 9B, for more information about this probe.)

What fraction of the shape is shaded?



Circle the answer

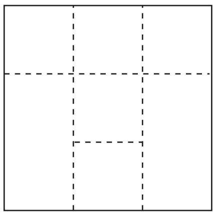
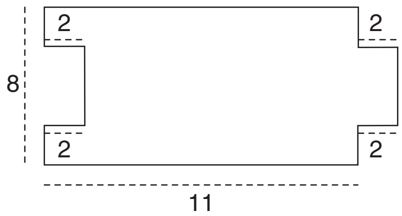
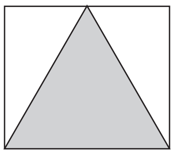
A) $\frac{1}{4}$

B) $\frac{1}{3}$

C) $\frac{1}{2}$

2. Multiple selections response. Two or more sets of problems, each with one question, one correct answer, and one or more distracters per problem (see Figure 1.9).

Figure 1.9 What's the Area? (See page 174, Probe 22, for more information about this probe.)

Item	Select Answer
A)  Explain your thinking:	Area of Rectangle? a) 12 sq units b) 6 sq units c) 9 sq units d) 5 sq units e) Not enough Information to find area
B)  Explain your thinking:	Area of the Figure? a) 88 sq units b) 27 sq units c) 38 sq units d) Not enough Information to find area
C)  Area of Triangle = 7 sq units Explain your thinking:	Area of Rectangle? a) 49 sq units b) 14 sq units c) 28 sq units d) 21 sq units e) Not enough Information to find area

3. Open response. One or more sets of items, each with one question. The open-response format does not include Tier 1 selected response choices on the student probe (see Figure 1.10). Typical incorrect responses are provided in the teacher's notes rather than listed as distracters.

Figure 1.10 How Many Stars? (See page 42, Probe 1, for more information about this probe.)

(Student Interview Task)

Student writes answer on line.

How many stars are there?

__________ 

Follow the directions below once student writes response.

Teacher points to the digit in the ones place of the student response and says,

"Can you use this *red* pencil to circle how many stars this number means from your answer?"

Next, point to the digit in the tens place of the student response and say,

Now use this *blue* pencil to circle how many stars this number means from your answer.

4. Opposing views/answers. Two or more statements are given, and students are asked to choose the statement they agree with (see Figure 1.11). This format is adapted from *Concept Cartoons in Science Education,* created by Stuart Naylor and Brenda Keogh (2000), for probing student ideas in science.

Figure 1.11 Quadrilaterals (See page 166, Probe 21, for more information on this probe.)

Circle the name or names of the people you agree with.

Tom	Jack	Pam	Sue	Mike
A square is always a rectangle.	A rectangle is sometimes a rhombus.	A rhombus is always a square.	A rectangle is sometimes a square.	A parallelogram is always a rhombus.

5. Examples and nonexamples list. One question or statement with several examples and nonexamples pertaining to a statement listed below. Students are asked to find only the examples based on a given statement (see Figure 1.12). This probe structure is often set up as a card sort where students are given one problem per card and asked to sort the problems into 2 piles.

Figure 1.12 Equal to 4? (See page 78, Probe 7, for more information on this probe.)

Circle only the math sentences where $\square = 4$.

A) $2 + 2 = \square - 3$	D) $\square = 1 + 3$
B) $9 - \square = 5$	E) $6 + 3 = \square + 5$
C) $10 - 6 = \square$	F) $3 + 1 = \square + 2$

6. Justified list. Two or more separate questions or statements are given, and students are asked to explain each choice.

Figure 1.13 Is It Equivalent? (See page 104, Probe 11, for more information on this probe.)

Equivalent to $\frac{2}{5}$?		Explain why or why not.
A) 2.5	Yes No	
B) 25%	Yes No	
C) 0.4	Yes No	
D) 0.25	Yes No	
E) 40%	Yes No	
F) 2.5%	Yes No	
G) 0.04	Yes No	

7. Strategy elicitation. A problem is stated with multiple solution strategies given. Students provide an explanation regarding making sense of each strategy.

Tier 2: Elaboration

The second tier of each of the probes is designed for individual elaboration of the reasoning used to respond to the question asked in the first tier. Mathematics teachers gain a wealth of information by delving into the thinking behind students' answers not just when answers are wrong but also when they are correct (Burns, 2005). Although the Tier 1 answers and distracters are designed around common understandings and misunderstandings, the elaboration tier allows educators to

Figure 1.14 What's Your Multiplication Strategy? (See page 144, Probe 17, for more information on this probe.)

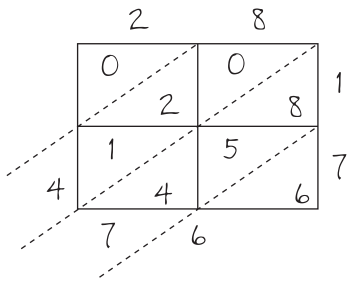
Sam, Julie, Pete, and Lisa each multiplied 18 by 17.

1. *Circle* the method that most closely matches how you solved the problem.
2. *Explain* whether *each* method makes sense mathematically.

A) Sam's Method

$$\begin{array}{r} {}^{5}28 \\ 17 \\ \hline {}^{1}196 \\ +\ 280 \\ \hline 476 \end{array}$$

B) Julie's Method



C) Pete's Method

$$28 \times 10 = 280$$
$$28 \times 5 = 140$$
$$28 \times 2 = 56$$
$$280 + 140 + 56 = 476$$

D) Lisa's Method

	20	8
10	200	80
7	140	56

$$200 + 80 + 140 + 56 = 476$$

look more deeply at student thinking as sometimes a student chooses a specific response, correct or incorrect, for an atypical reason. Also, there are many different ways to approach a problem correctly; therefore, the elaboration tier allows educators to look for trends in thinking and in methods used.

Also important to consider is the idea that in order to address misconceptions, students must be confronted with their own incorrect ideas by participating in instruction that causes cognitive dissonance between existing ideas and new ideas. By having students complete both tiers of a probe and then planning instruction that addresses the identified areas of difficulty, teachers can then use students' original responses as part of a reflection on what was learned. Without this pre-assessment commitment of selecting an answer and explaining the choice, new understanding and corrected ideas are not always evident to the student.

WHAT ADDITIONAL INFORMATION IS PROVIDED WITH EACH MATHEMATICS ASSESSMENT PROBE?

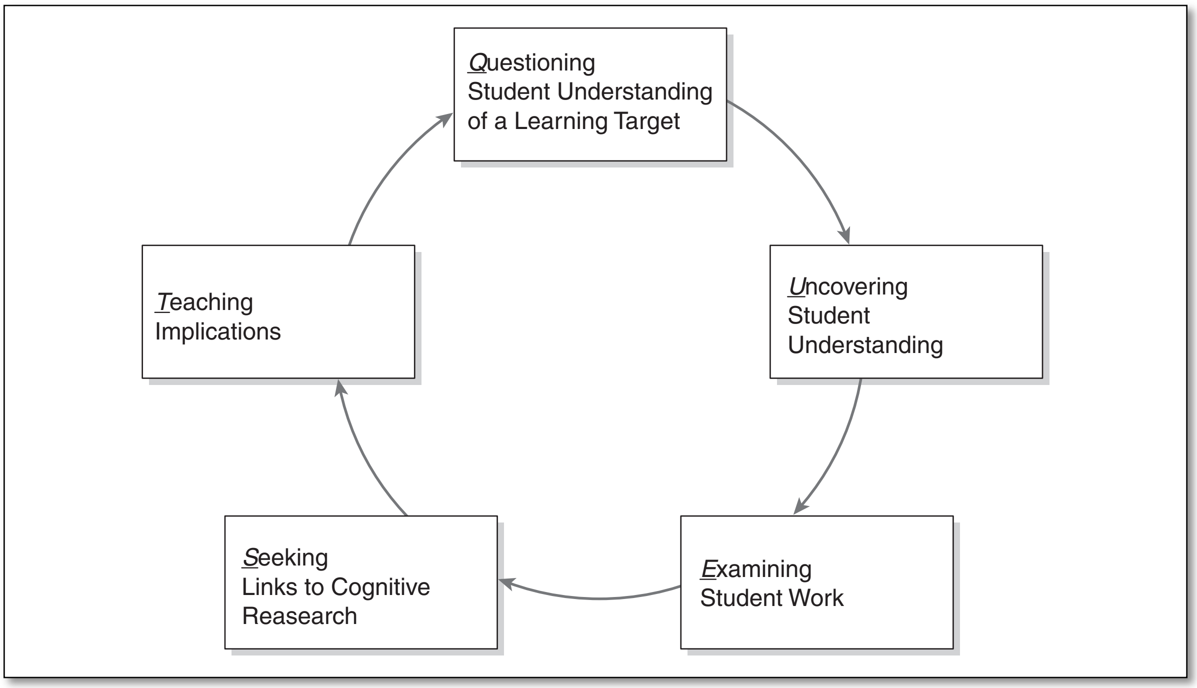
In *Designing Professional Development for Teachers of Science and Mathematics*, Loucks-Horsley, Love, Stiles, Mundry, and Hewson (2003) describe action research as an effective professional development strategy. To use the probes in this manner, it is important to consider the complete implementation process.

We refer to an action research quest as working through the full cycle of

- questioning student understanding of a particular concept;
- uncovering understandings and misunderstandings using a probe;
- examining student work;
- seeking links to cognitive research to drive next steps in instruction; and
- teaching implications based on findings and determining impact on learning by asking an additional question.

The Teacher Notes, included with each probe, have been designed around the action research QUEST cycle, and each set of notes includes relevant information for each component of the cycle (see Figure 1.15). These components are described in detail below.

Figure 1.15 QUEST Cycle



Source: Rose, C., Minton, L., & Arline, C. (2007).

Questioning for Student Understanding

This component helps to focus a teacher on what a particular probe elicits and to provide information on grade-appropriate knowledge. Figure 1.16 shows an example question from the Mathematics Assessment Probe, What's Your Addition Strategy?

Figure 1.16 Sample Question From the What's Your Addition Strategy? Probe

Questioning for Student Understanding

When adding, can students apply and understand a variety of different strategies?

K–2	*3–5*	

Grade-span bars are provided to indicate the developmentally appropriate level of mathematics as aligned to the NCTM Standards and cognitive research. The dark gray band represents the grade levels where the mathematics required of the probe is aligned to the standards, and the lighter gray band shows grade levels where field testing of the probe has indicated students still have difficulties. The grade spans, although aligned to the standards, should be considered benchmarks as some students at higher grades may have misunderstandings based in understandings from lower grades, while others may be further along the learning progression and need probes designed for older students.

Uncovering Understanding

Figure 1.17 shows an example, Uncovering Understanding, from the Mathematics Assessment Probe, What's Your Addition Strategy?

Following the Teacher Notes and sample student responses, adaptations or variations of the Mathematics Assessment Probe are provided for some of the probes. Variations of the probes provide a different structure (selected response, multiple selections, opposing views, examples/nonexamples, justified list, and strategy harvest) for the question within the same grade span. An adaptation to the probe is similar in content to the original, but the level of mathematics changes for use with a different grade span. When a variation and/or adaptation of the probe is provided, the information is included in the Uncovering Understanding section.

Figure 1.17 Uncovering Understanding

Uncovering Understanding

Addition Strategies: Whole Numbers Content Standard: Number and Operation

Important Note: Prior to giving students the probe, ask them to individually solve the indicated problem. If time allows, ask them to solve the problem in at least two or three different ways.

Variations/Adaptations:

- Addition Strategies: Three-Digit Numbers
- Addition Strategies: One-Digit Numbers

Examining Student Work

This section includes information about the stem, answers, and distracters as related to the research on cognitive learning. Example student responses are given for a selected number of elicited understandings and misunderstandings. The categories, conceptual/procedural and common errors/overgeneralizations, are used where appropriate and are written in italics. Figure 1.18 shows an example of Examining Student Work, from the Mathematics Assessment Probe, What's Your Addition Strategy?

Figure 1.18 Sample of Examining Student Work From the What's Your Addition Strategy? Probe

Examining Student Work

Student answers may reveal *misunderstandings* regarding methods of addition, including a lack of *conceptual understanding* of number properties. Responses also may reveal a common misconception that there is only one correct algorithm for each operation or that, once comfortable with a method, there is no need to understand other methods.

- Sam's method: This method is usually recognized by third-to-fifth-grade students, although in some situations students may not have been introduced to this standard U.S. algorithm. Those who have no experience with the method show lack of *procedural understanding* of the algorithm and typically indicate the method "does not make sense because 1 + 34 + 56 is 91, not 90." Those students who do recognize the algorithm often do not demonstrate place value understanding. (See Student Responses 1 and 2, Chapter 5, page 133.)
- Julie's method: This method is often recognized by students who have experience with multiple algorithms as well as those who were taught only the traditional algorithm. These latter students often apply variations using an expanded notation form of the numbers. (See Student Response 3, Chapter 5, page 133.)
- Pete's method: This strategy is often the least recognized by students in terms of generalizing a method of adding and subtracting like amounts from the numbers to keep a constant total. (See Student Response 4, Chapter 5, page 133.)
- Lisa's method: Students who use this strategy hold the first number constant and break the addend into place value parts, adding on one part at a time. Students who have experience using an open number line are typically able to mathematically explain this method of addition. (See Student Response 5, Chapter 5, page 133.)

Seeking Links to Cognitive Research

This section provides additional information about research that teachers can use for further study of the topic. Figure 1.19 shows an example from the Mathematics Assessment Probe, What's Your Addition Strategy?

Figure 1.19 Seeking Links to Cognitive Research

Seeking Links to Cognitive Research

Student errors when operating on whole numbers suggest students interpret and treat multi digit numbers as single-digit numbers placed adjacent to each other, rather than using place-value meanings for digits in different positions. (AAAS, 1993, p. 358)

The written place-value system is a very efficient system that lets people write very large numbers. Yet it is very abstract and can be misleading: The digits in every place look the same. To understand the meaning of the digits in the various places, children need experience with some kind of *size-quantity supports* (e.g., objects or drawings) that show tens to be collections of 10 ones and show hundreds to be simultaneously 10 tens and 100 ones, and so on. (NCTM, 2003, p. 78)

Students can use roughly three classes of effective methods for multi digit addition and subtraction, although some methods are mixtures. *Counting list methods* are extensions of the single-digit counting methods. Children initially may count large numbers by ones, but these unitary methods are highly inaccurate and are not effective. All children need to be helped as rapidly as possible to develop prerequisites for methods using tens. These methods generalize readily to counting on or up by hundreds but become unwieldy for larger numbers. In *decomposing methods,* children decompose numbers so that they can add or subtract the like units (add tens to tens, ones to ones, hundreds to hundreds, etc.). These methods generalize easily to very large numbers. *Recomposing methods* are like the make-a-ten or doubles methods. The solver changes both numbers by giving some amount of one number to another number (i.e., in adding) or by changing both numbers equivalently to maintain the same difference (i.e., in subtracting). (NCTM, 2003, p. 79)

When students merely memorize procedures, the may fail to understand the deeper ideas. When subtracting, for example, many children subtract the smaller number from the larger in each column, no matter where it is. (National Research Council, 2002b, p. 13)

By the end of the 3–5 grade span, students should be computing fluently with whole numbers. *Computational fluency* refers to having efficient and accurate methods for computing. Students exhibit computational fluency when they demonstrate flexibility in the computational methods they choose, understand and can explain these methods, and produce accurate answers efficiently. The computational methods that a student uses should be based on mathematical ideas that the student understands well, including the structure of the base-ten number system. (NCTM, 2000, p. 152)

Computation skills should be regarded as tools that further understanding, not as a substitute for understanding. (Paulos, 1991, p. 53)

When students merely memorize procedures, they may fail to understand the deeper ideas that could make it easier to remember—and apply—what they learn. Understanding makes it easier to learn skills, while learning procedures can strengthen and develop mathematical understanding. (NRC, 2002, p. 13)

Study results indicate that almost all children can and do invent strategies and that this process of invention (especially when it comes *before* learning standard algorithms) may have multiple advantages. (NCTM, 2002c, p. 93)

Teaching Implications

Being aware of student difficulties and their sources is important, but designing instruction is most critical in helping diminish those difficulties. Although some ideas are included, through "Focus Through Instruction" statements and "Questions to Consider," the authors strongly encourage educators to use the *Curriculum Topic Study (CTS)* (Keeley & Rose, 2006) process to search for additional teaching implications. Each set of Teacher Notes refers the related *CTS* guide for further study, additional references, and a Teacher Sound Bite from a field tester of the probe. Figure 1.20 shows an example from the Mathematics Assessment Probe, What's Your Addition Strategy?

Figure 1.20 Teaching Implications

Focus Through Instruction

- Focusing on understanding multidigit addition methods results in much higher levels of correct use of methods.
- Students need visuals to understand the meanings of hundreds, tens, and ones. These meanings need to be related to the oral and written numerical methods developed in the classroom.
- Number lines and hundreds grids support counting-list methods the most effectively.
- Decomposition methods are facilitated by objects that allow children to physically add or remove different quantities (e.g., base-10 blocks).
- Student's who believe there are several correct methods for adding numbers often show higher engagement levels.
- When children solve multidigit addition and subtraction problems, two types of problem-solving strategies are commonly used: invented strategies and standard algorithms. Although standard algorithms can simplify calculations, the procedures can be used without understanding, and multiple procedural issues can exist.
- Both invented and standard algorithms can be analyzed and compared, helping students understand the nature and properties of the operation, place value concepts for numbers, and characteristics of efficient methods and strategies.

Questions to Consider (when working with students as they develop and/or interpret a variety of algorithms)

- When exploring and/or inventing algorithms, do students consider the generalizability of the method?
- Are students able to decompose and recompose the type of number they are operating with?
- Can students explain why a strategy results in the correct answer?
- When analyzing a strategy or learning a new method, do students focus on properties of numbers and the underlying mathematics rather than just memorizing a step-by-step procedure?
- Do students use a variety of estimation strategies to check the reasonableness of the results?

Teacher Sound Bite

I struggle to know what methods students bring with them each year when transitioning to my fourth-grade class. These strategy probes help me consider my students' level of comfort and experience with a variety of ways to add numbers and whether they have more than just an understanding of the steps a procedure.

(Continued)

Figure 1.20 (Continued)

Curriculum Topic Study and Uncovering Student Thinking

Place Value

What's Your Addition Strategy? (page 128)

Keeley, P., & Rose, C. (2007). *Mathematics curriculum topic study: Bridging the gap between standards and practice.* Thousand Oaks, CA: Corwin. (Addition and Subtraction, p. 111).

Related Elementary Probes:

Rose, C., & Arline, C. (2009). *Uncovering student thinking in mathematics, grades 6–12: 30 formative assessment probes for the secondary classroom.* Thousand Oaks, CA: Corwin. (Variation: What's Your Addition Strategy? Decimals, p. 82; Fractions, p. 83).

Additional References for Research and Teaching implications

McREL. (2002). *EDThoughts: What we know about mathematics teaching and learning.* Bloomington, IN: Solution Tree. (pp. 82–83).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author. (p. 152).

National Council of Teachers of Mathematics. (2002). Lessons learned from research. Reston, VA: Author. (pp. 93–100).

National Council of Teachers of Mathematics. (2003). *Research companion to principles and standards for school mathematics.* Reston, VA: Author. (pp. 68–84).

National Research Council. (2002). *Helping children learn mathematics.* Washington, DC: National Academy Press. (pp. 11–13).

National Research Council. (2005). *How students learn: Mathematics in the classroom.* Washington, DC: National Academy Press. (pp. 223–231).

Paulos, J. A. (1991). *Beyond numeracy.* New York: Vintage. (pp. 52–55).

A note about the use of interactive technology applets: Some of the concepts elicited by the probes can be addresed through available online resources. Keep in mind that most of these applets were developed as instructional resources or to provide practice and were not developed to address a specific misconception. When searching for available applets that meet students' needs as elicited by a probe, be sure to review the applet carefully to consider the range of examples and nonexamples that can be modeled using the tool. Before using with students, prepare a scaffolded set of questions designed specifically to highlight the misunderstandings elicited by the probe items.

Following is a sample list of sites to look at for freely available interactive applets:

- National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html)
- NCTM's Illuminations (http://illuminations.nctm.org)
- Educational Development Center (http://maine.edc.org/file.php/1/K6.html)

In addition to the Teacher Notes, a Note Template is included in the back of the book (see Resource: Notes Template: QUEST Cycle). The Note Template provides a structured approach to working through a probe quest. The components of the template are described in Figure 1.21.

Figure 1.21 Notes Template: QUEST Cycle

_Q_uestioning for Student Understanding of a Particular Concept

Considerations: What is the concept you wish to target? Is the concept at grade level or is it a prerequisite?

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_U_ncovering Understandings and Misunderstandings Using a Probe

Considerations: How will you collect information from students (paper-pencil responses, interview, student response system, etc.)? What form will you use (one-page probe, card sort, etc.)? Are there adaptations you plan to make? Review the summary of typical student responses. What do you predict to be common understandings and/or misunderstandings for your students?

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_E_xamining Student Work

Sort by selected responses then re-sort by trends in thinking
Considerations: What common understanding and misunderstandings were elicited by the probe?

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_S_eeking Links to Cognitive Research to Drive Next Steps in Instruction

Considerations: How do these elicited understanding and misunderstandings compare to those listed in the Teacher Notes? Review the bulleted items in the Focus Through Instruction and Questions to Consider to begin planning next steps. What additional sources did you review?

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_T_eaching Implications Based on Findings and Determining Impact on Learning by Asking an Additional Question

Considerations: What actions did you take? How did you assess the impact of those actions? What are your next steps?

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WHAT MATHEMATICS ASSESSMENT PROBES ARE INCLUDED IN THE BOOK?

Many of the samples included in this book fall under numerical concepts and operations because the cognitive research is abundant in these areas at Grades K–5. The book also includes multiple examples for the following additional content standards: algebra, data analysis, geometry, and measurement. Figure 1.22 provides an "at a glance" look of the grade span and content of the 25 probes with Teacher Notes that are included in Chapters 3 through 6. Grade-span bars are provided to indicate the developmentally appropriate level of mathematics as aligned to NCTM Standards as well as the cognitive research.

Figure 1.22 Table of Probes With Teacher Notes

Grade-Span Bar Key

(dark shading)	Target for Instruction Depending on Local Standards
(light shading)	Prerequisite Concept and Field Testing Indicate Students May Have Difficulty

Question	Probe	Grade Span					
Structure of Number: Place Value, Number Charts, and Number Lines		K	1	2	3	4	5
Do students understand the value of each digit in a double-digit number?	How Many Stars? (page 42)		1–2		3		
Can students translate numbers from verbal to symbolic representation?	What's the Number? (page 49)		1–2		3		
Can students choose all correct values of various digits of a given whole number?	What Is the Value of the Place? (page 54)		1–3			4	
Are students able to estimate a value on a number line given the value of the endpoints?	What Number Is That? (page 58)		1–2		3–4		
Given a portion of the 100s chart, can students correctly fill in the missing values?	Hundred Chart Chunks (page 64)			2–3		4–5	
Can students choose all correct values of various digits of a given decimal?	What Is the Value of the Place? (page 71)				3–4		5
Structure of Number: Parts and Wholes							
Do students understand the meaning of the equal sign?	Equal to 4? (page 78)			2–4			5
Given a set model, are students able to define fractional parts of the whole?	Crayon Count (page 85)			2–4			5
Given a whole, are students able to indentify when $\frac{1}{4}$ of the whole is shaded?	Is $\frac{1}{4}$ of the Whole Shaded? (page 91)				3–4		5
Given an area model, are students able to define fractional parts of the whole?	Granola Bar (page 99)					4–5	
Are students able to choose equivalent forms of a fraction?	Is It Equivalent? (page 104)					4–5	
Do students use the "cancelling of zeros" shortcut appropriately?	Is It Simplified? (page 110)					4–5	

Question	**Probe**	**Grade Span**					
Structure of Number: Computation and Estimation		K	1	2	3	4	5
Do students use the structure of ten when combining collections?	How Many Dots? (page 116)	K	1–2				
When considering the whole and two parts, can students identify all possible part-part-whole combinations?	Play Ball (page 122)		1–2		3		
When adding, can students apply and understand a variety of different strategies?	What's Your Addition Strategy? (page 128)			2–3		4–5	
When subtracting, can students apply and understand a variety of different strategies?	What's Your Subtraction Strategy? (page 136)			2–3		4–5	
Are students flexible in using strategies for solving various multiplication problems?	What's Your Multiplication Strategy? (page 144)					4–5	
Are students flexible in using strategies for solving division problems?	What's Your Division Strategy? (page 150)					4–5	
Do students understand there are multiple methods of estimating the sum of three 2-digit numbers?	Is It an Estimate? (page 155)					4–5	
Can students use estimation to choose the closest benchmark to an addition problem involving fractions?	What Is Your Estimate? (page 160)					4–5	
Measurement and Geometry							
Do students understand the properties and characteristics of quadrilaterals?	Quadrilaterals (page 166)				3–4		5
Are students able to determine area without the typical length by width labelling?	What's the Area? (page 174)				3–5		
Do students pay attention to starting point when measuring with nonstandard units?	What's the Measure? (page 180)				3–5		
Data							
Are students able to choose the correct graphical representation when given a mathematical situation?	Graph Choices (page 187)				3–5		
Do students understand ways the median is affected by changes to a data set?	The Median (page 194)				3–5		

In addition to the 25 probes with teacher notes, many variations to the probes are provided. Although some of the variations provide a different structure option, others are created to target a different grade span by addressing a foundational concept to the original probe. For these foundational variations to the probe, information from the Teacher Notes, although helpful, is not intended to align directly. Therefore we strongly suggest you use the QUEST cycle template to create your own version of teacher notes. The following table provides a grade-level view of the probes and variations that are included in this resource.

Figure 1.23 Table of Probes by Targeted Grade Span

Grade K		
Probe	*Grade Span*	*Chapter/Page*
Variation: How Many Counters?	K	Chapter 5, page 121
How Many Dots?	K	Chapter 5, page 116
Variation: What's the Measure?	K–1	Chapter 6, page 186
Variation: Is It a Circle?	K–1	Chapter 6, page 173
Grade 1		
Probe	*Grade Span*	*Page*
What Is the Value of the Place?	1–2	Chapter 3, page 54
What's the Number?	1–2	Chapter 3, page 49
What Number Is That?	1–2	Chapter 3, page 58
How Many Stars?	1–2	Chapter 3, page 42
Variation: How Many Counters?	1	Chapter 5, page 121
Play Ball	1–2	Chapter 5, page 122
Variation: What's the Measure?	K–1	Chapter 6, page 186
Variation: Is It a Circle?	K–1	Chapter 6, page 173
Grade 2		
Probe	*Grade Span*	*Page*
What's the Number?	1–2	Chapter 3, page 49
What Is the Value of the Place?	2–3	Chapter 3, page 54
What Number Is That?	1–2	Chapter 3, page 58
Hundred Chart Chunks	2–3	Chapter 3, page 64
Equal to 4?	2–4	Chapter 4, page 78
Crayon Count	2–4	Chapter 4, page 85
Play Ball	1–2	Chapter 5, page 122
What's Your Addition Strategy?	2–3	Chapter 5, page 128
What's Your Subtraction Strategy?	2–3	Chapter 5, page 136
Variation: Is It a Polygon?	2–3	Chapter 6, page 172
Grade 3		
Probe	*Grade Span*	*Page*
Hundred Chart Chunks	2–3	Chapter 3, page 64
What is the Value of the Place?	2–3	Chapter 3, page 54
What Is the Value of the Digit?	3–5	Chapter 3, page 71
Equal to 4?	2–4	Chapter 4, page 78
Crayon Count	2–3	Chapter 4, page 85
Is $\frac{1}{4}$ of the Whole Shaded?	3–4	Chapter 4, page 91

Probe	*Grade Span*	*Page*
What's Your Addition Strategy?	2–3	Chapter 5, page 128
What's Your Subtraction Strategy?	2–3	Chapter 5, page 136
Quadrilaterals	3–4	Chapter 6, page 166
Variation: What's the Measure?	3–5	Chapter 6, page 186
What's the Area?	3–5	Chapter 6, page 174
Graph Choices	3–5	Chapter 6, page 187
The Median	3–5	Chapter 6, page 194
Variation: Is It a Polygon?	2–3	Chapter 6, page 172
Grade 4		
Probe	*Grade Span*	*Page*
What Is the Value of the Digit?	3–5	Chapter 3, page 71
Equal to 4?	2–4	Chapter 4, page 78
Is $\frac{1}{4}$ of the Whole Shaded?	3–4	Chapter 4, page 91
Granola Bar	4–5	Chapter 4, page 99
Is It Equivalent?	4–5	Chapter 4, page 104
Is It Simplified?	4–5	Chapter 4, page 110
Is It an Estimate?	4–5	Chapter 5, page 155
What's Your Multiplication Strategy?	4–5	Chapter 5, page 144
What's Your Division Strategy?	4–5	Chapter 5, page 150
What Is Your Estimate?	4–5	Chapter 5, page 160
Quadrilaterals	3–4	Chapter 6, page 166
What's the Measure?	3–5	Chapter 6, page 180
Variation: Length of a Line	4–5	Chapter 6, page 185
What's the Area?	3–5	Chapter 6, page 174
Graph Choices	3–5	Chapter 6, page 187
The Median	3–5	Chapter 6, page 194
Grade 5		
Probe	*Grade Span*	*Page*
What Is the Value of the Digit?	3–5	Chapter 3, page 71
Granola Bar	4–5	Chapter 4, page 99
Is It Equivalent?	4–5	Chapter 4, page 104
Is It Simplified?	4–5	Chapter 4, page 110
Is It an Estimate?	4–5	Chapter 5, page 155
What's Your Multiplication Strategy?	4–5	Chapter 5, page 144
What's Your Division Strategy?	4–5	Chapter 5, page 150
What Is Your Estimate?	4–5	Chapter 5, page 160
What's the Measure?	3–5	Chapter 6, page 180
Variation: Length of a Line	4–5	Chapter 6, page 185
What's the Area?	3–5	Chapter 6, page 174
Graph Choices	3–5	Chapter 6, page 187
The Median	3–5	Chapter 6, page 194